MICHAEL DETLEFSEN

POINCARÉ AGAINST THE LOGICIANS*

ABSTRACT. Poincaré was a persistent critic of logicism. Unlike most critics of logicism, however, he did not focus his attention on the basic laws of the logicists or the question of their genuinely logical status. Instead, he directed his remarks against the place accorded to logical *inference* in the logicist's conception of mathematical proof. Following Leibniz, traditional logicist dogma (and this is explicit in Frege) has held that reasoning or inference is everywhere the same – that there are no principles of inference specific to a given local topic. Poincaré, a Kantian, disagreed with this. Indeed, he believed that the use of non-logical reasoning was essential to genuinely mathematical reasoning (proof). In this essay, I try to isolate and clarify this idea and to describe the mathematical epistemology which underlies it. Central to this epistemology (which is basically Kantian in orientation, and closely similar to that advocated by Brouwer) is a principle of epistemic conservation which says that knowledge of a given type cannot be extended by means of an inference unless that inference itself constitutes knowledge belonging to the given type.

1.

In the philosophy of mathematics, Poincaré is probably best known for his disagreement with the logicists – in particular, with Russell, with whom he carried on a running debate in the early years of this century.[1] Unlike the usual criticisms of logicism, however, Poincaré's critique did not focus on the question of the status of the 'basic laws' of the logicist's systems, finding difficulties in seeing them as genuinely *logical* principles. Rather, it was dominated by the quite different idea that there is little, if any, place for logical *inference* in mathematical proof – such inferences being epistemically too colorless to be a part of any genuinely mathematical reasoning. Logical inference, by its very nature, applies everywhere, and so neither requires nor reflects any distinctively *mathematical* knowledge; and, for that reason, it cannot be expected to serve as means of *extending* genuinely mathematical knowledge.

Poincaré's defense of these views rested on an appeal to what he took to be a datum of mathematical 'common sense'. Anyone with mathematical experience would, he maintained, clearly perceive a large and important difference between the epistemic condition of one whose reasoning is based on the topic-blind steps of logical inference (e.g.,

Synthese **90**: 349–378, 1992.

modus ponens and the like), and one whose reasoning is based on topic-specific penetration of a particular mathematical subject. The mathematician's inferences stem from and reflect a knowledge of the local "architecture" (Poincaré's term) of the particular subject with which they are concerned, while those of the logician represent only a globally valid, topic-neutral (and, therefore, locally insensitive!) form of knowledge.[2] Using Poincaré's own figure, the "logician" is like a writer who is well-versed in grammar, but has no ideas.[3]

This, in brief, is Poincaré's view of the place of logic in mathematical reasoning. But though interesting and distinctive, and an absolutely central part of Poincaré's philosophy of mathematics, it has never been developed in any systematic way. The chief task of this paper is to take steps toward remedying this deficiency, with the hope that, the philosophical basis of Poincaré's views having been more clearly formulated, their interest might be more deeply appreciated, and their plausibility and importance more accurately judged.

The views offered here differ sharply with other recent attempts – most notably, that by Warren Goldfarb in 'Poincaré Against the Logicists'[4] – to elaborate Poincaré's philosophical ideas. Goldfarb claims that what marks Poincaré's conception of foundations and sets it apart from the rival conception of the logicists is its concern with providing a psychologically realistic account of mathematical knowledge and reasoning; that is, an account of mathematical knowledge which describes, in a psychologically realistic way, how it is that we come to have it.

This attempt to make psychological plausibility the focus of the dispute between Poincaré and the logicists is tempting not least because the preeminent logicists, Frege and Russell (and, to a lesser extent, Cantor), are well known for their anti-psychologistic bent in foundations. Though quite different from one another, they nonetheless both held views which agree in seeing the task of mathematical epistemology as set by metaphysics rather than psychology. In Russell's case it is a metaphysics of objects and propositions (cf. Russell 1903, p. 427). In Frege's case it is a metaphysical ordering of truths, which is seen as capturing the relations of (objectively metaphysical) "sufficient reason" obtaining among them.[5] In neither case is psychological plausibility anything but a 'red herring'; mathematical epistemology, on their views, is properly obligated to facts of mathematical being, not to facts of mathematical believing.

But though there are serious differences between the logicist and

Poincaréan conceptions of the foundational enterprise (some of the more important of which will be discussed in what follows), we do not see these differences as centering on the question of whether foundations should be psychologically realistic.[6] Thus, we disagree with Goldfarb's interpretation of Poincaréan foundations. As we see it, Poincaré's viewpoint is characterized not by a concern for the *psychological* differences between the Poincaréan prover and his 'logician' counterpart but, rather, by a concern for their *epistemological* differences. Hence, the issue is not whether a mathematical epistemology yields a psychologically accurate description of the mathematical knower but, rather, whether it provides a plausible account of the epistemological differences separating the knowledge of the mathematical knower from that of the 'logician'. Such, at any rate, is the view that we will be defending in this essay.

Structurally, what we end up with is an epistemology centering on two complementary principles. One of these is a *principle of epistemic typification* according to which knowledge is to be seen as divided into types, but types whose division does not follow the divisions between subject-matters that are commonly used to demarcate mathematics from non-mathematics.[7] The other is a *principle of epistemic conservation* which says, in effect, that inference cannot be *epistemically* 'creative' – that is, that it cannot give rise to knowledge belonging to a given type unless it itself constitutes knowledge of that type. In the case of mathematical knowledge, this second principle implies that if reasoning to a mathematical conclusion from a group of premises is to be capable of producing mathematical knowledge of that conclusion, then, in addition to the premises being mathematically known, it must also be the case that the inferences used in conducting the proof themselves constitute *mathematical* knowledge. In Poincaré's view, it is this latter condition that is violated by the use of purely *logical* inference in mathematical reasoning.

Adopting such a view concerning the place of logical inference in mathematical proof has serious consequences, among which the clash with standard views of rigor is among the more noticeable and important. One of the celebrated triumphs of the 'logicization' of mathematical reasoning by Frege, Peano, and Russell and Whitehead was the new and seemingly unimpeachable standard of rigor that it brought with it; a standard which continues to guide foundational work to the present day. On this approach, a rigorous proof is one in which all substantive (i.e., topic-specific) information has been driven out of the inferences and into the axioms, thence to be explicitly registered in the premises of the proofs in which it is used. The end result – or such, at any rate, is the pious hope – is a clear and accurate record of the beliefs upon which a given conclusion is based.

This end is, however, achieved only by reducing all inferences to *logical* ones since it is only such inferences which, being topic-neutral, cannot themselves contain any substantive (i.e., topic-specific) information. Logic-sized steps of inference are thus exactly what rigor seems to demand, since it is only through using such steps that all substantive information used in a given proof is forced to be explicitly declared in the premises.

This traditional conception of rigor thus depends on there being globally valid, topic-neutral forms of inference in terms of which the inferences pertaining to a given subject can be cast. Without this, we run the risk of smuggling undeclared – and, therefore, unrecognized – information into our proofs. The Poincaréan is thus challenged to show how rigor can be achieved when the globally valid, topic-neutral inferences of logic are banished and replaced with the locally valid, topic-specific inferences of the 'mathematician'.

His response is as radical as it is simple. Rigor will be achieved not by the elimination of logical or informational gaps separating premises from conclusions (hence, the elimination of substantive inference) but, rather, by the elimination of gaps *in our mathematical understanding* (and, hence, the elimination of inferences in which the premises do not constitute a good *mathematical* reason for the conclusion).[8] Seen in this way, an inference is rigorous when, and only when, it reflects genuine mathematical understanding; and such a view is the very antithesis of the standard one, since it requires inference to be substantive *rather than* logical in order to be rigorous.

This then, in outline, is the argument of the present paper. Its chief objectives are the elaboration of the ideas which we take to underlie the Poincaréan viewpoint in the philosophy of mathematics.

2. THE 'INSOLUBLE CONTRADICTION' OF MATHEMATICAL KNOWLEDGE

In a manner reminiscent of Kant's opening remarks to The First Part of the Transcendental Problem of the *Prolegomena*,[9] Poincaré opens *Science and Hypothesis* with these words:

The very possibility of the science of mathematics seems an insoluble contradiction. If this science is deductive only in appearance, whence does it derive that perfect rigor no one dreams of doubting? If, on the contrary, all the propositions it enunciates can be deduced from one another by the rules of normal logic, why is not mathematics reduced to an immense tautology? The syllogism can teach us nothing essentially new, and, if everything is to spring from the principle of identity, everything should be capable of being reduced to it. Shall we then admit that the enunciations of all those theorems which fill so many volumes are nothing but devious ways of saying A is A? (Poincaré 1902, p. 31)

There are, however, some important differences between Kant and Poincaré. Both recognize what might be called the 'epistemic substantiveness' of mathematics (i.e., the fact that it constitutes a significant and substantial body of knowledge) as a datum for mathematical epistemology. However, while for Kant it is the "apodeictic certainty" of mathematics that is presented as the competitor of epistemic substantiveness, for Poincaré it is its "perfect rigor".

A closer reading, however, raises the possibility that Poincaré was not so much intending to pose a dilemma (between epistemic substantiveness and perfect rigor) for mathematical epistemology generally as he was to give a critique of one particular form that such an epistemology had taken; namely, that of a Leibniz-style logicism.[10] For though what Poincaré presents as contradictory are the claims that

(I) Mathematics is perfectly rigorous,

and

(II) Mathematical theorems are not merely logical truths or tautologies,

these claims are contradictory (even loosely speaking) only if (I) is taken to imply something which it clearly does not; namely, that the theorems of mathematics are all tautologies. In truth, what (I) seems to require is not that all *theorems* of mathematics be *logical truths* but, rather, that all *inferences* in a mathematical proof be *logical inferences*. Thus, the second question which Poincaré asks in the above-quoted remark (viz., "If all the propositions it enunciates can be deduced from one another by the rules of formal logic, why is not mathematics reduced to an immense tautology?"), which is supposed to have no (ready) answer, would actually appear to have an easy one: namely, "Because the *axioms* with which the deductions begin are not themselves logical truths". It would not, therefore, appear to have the force that Poincaré wanted it to have.

Is Poincaré's 'insoluble contradiction' based on a simple-minded failure to recognize this elementary point? The paragraph directly succeeding the one quoted above suggests that this is not so.[11]

Without doubt, we can go back to the axioms, which are the source of all these reasonings. If we decide that these cannot be reduced to the principle of contradiction, if still less we see in them experimental facts which could not partake of mathematical necessity, we have yet the resource of classing them among synthetic *a priori* judgements. This is not to solve the difficulty, but to baptize it; and even if the nature of synthetic judgements were for us no mystery, the contradiction would not have disappeared, it would only have moved back; syllogistic reasoning remains incapable of adding anything to the data given it; these data reduce themselves to a few axioms, and we should find nothing else in the conclusions.

It was thus clear to Poincaré that the inference from the given fact that mathematics is perfectly rigorous to the further assertion that mathematical theorems are tautologies can be blocked by adopting the position, open to non-logicists if not to logicists, that the axioms are not tautologies. Still, he insists, the problem he has in mind would not be avoided by such a move. This is so, he explains, because even if (contrary to logicism) it were granted that the axioms are not logical truths, one still faces the problem of explaining how the *theorems* of mathematics could constitute a genuine extension of the *axioms* if the only principles of inference used are such as have a purely logical character. Poincaré's contention, then, is that, if only purely logical inferences are used in a proof, knowledge thus gained of the theorem proven cannot constitute extensions of whatever *mathematical* knowledge might be represented by one's knowledge of the axioms used to prove it. Yet, despite this, he believed that the conclusions of *mathematical* proofs typically *do* represent epistemic extensions of their premises. Consequently, he was led to conclude that not all the inferences belonging to a typical mathematical proof can be of a purely logical character.

Where Poincaré saw a genuine contradiction, then, is between the following two principles.

(I′) All the inferences used in mathematical proof are of a purely logical character,

and

(II′) Conclusions of mathematical proofs can, and often do, constitute extensions of the mathematical knowledge represented by the premises.

The conflict between (I′) and (II′) gives rise to a like conflict between (I) and (II′) only if (I) is taken to imply (I′). This, however, Poincaré denied (cf. Poincaré 1908, ch. III (p. 385 in the Halsted translation)), recommending instead an alternative conception of mathematical rigor according to which it consists in the lack of 'gaps' in mathematical understanding. One need have no fear of missing elements in a mathematical inference provided that one grasps the mathematical reason behind it. Mathematical reasoning can thus proceed in larger-than-logic-sized steps to the extent that mathematical understanding permits it to.

Adopting this non-logical conception of rigor[12] allowed Poincaré to retain (I), which he held to be true, while rejecting (I′), which he held to be false. According to Poincaré, the falsehood of (I′) is evident from the fact that, were it to be true

> no theorem could be new if no new axiom intervened in its demonstration; reasoning could give us only the immediately evident verities borrowed from direct intuition; it would be only an intermediary parasite, and therefore should we not have good reason to ask whether the whole syllogistic apparatus did not serve solely to disguise our borrowing. (Poincaré 1902, p. 31)

Since, however, mathematical reasoning can, and typically does, yield new knowledge *without* the intervention of new axioms, it follows that (I′) is false. Hence, mathematical reasoning must typically make use of inferences that are not purely logical in character. In his own words:

> Mathematical reasoning has of itself a sort of creative virtue and consequently differs from the syllogism. (Poincaré 1902, p. 32)

In Poincaré's view, then, the non-logical character of mathematical inference is necessary for the explanation of (II′), which he accepted as a datum for mathematical epistemology.

The apparently 'insoluble contradiction' of which Poincaré speaks in the opening passage of *Science and Hypothesis* is thus not to be taken as representing a conflict between (I) and (II), though this might at first sight appear to be the gist of his misstated question inquiring how mathematics could be anything but an immense tautology, if, as the logicists would have it, all the propositions of mathematics were deduced from one another by purely logical reasoning. Nor is it to be confused with the conflict between (I′) and (II′), which Poincaré believed to be entirely genuine and unresolvable. Rather, it is that which exists between (I) and (II′), both of which Poincaré held to be true, and whose compatibility he sought to make plain through his rejection of the 'logician's' conception of rigor in favor of one which emphasizes the importance of mathematical rather than logical acumen.

3. THE NATURE OF MATHEMATICAL PROOF

What Poincaré primarily was concerned to challenge, then, was not just logicism but, rather, the 'logicization' of mathematical proof – by which we mean the reduction of all inferences occurring within a mathematical proof to logical inferences.[13] He believed that there are distinctively mathematical forms of inference, of which perhaps the clearest and most important is mathematical induction.[14] He believed, moreover, that realizing this fact is the key to sustaining the Kantian observation concerning the epistemic 'creativity' of mathematical proof. These views brought him into sharp conflict with the logicists, who believed not only that such logicization is necessary for the optimal form of rigor, but also that rational thought is essentially homogeneous or non-modular (i.e., non-local) in character. Such a doctrine, of course, implies that mathematical reasoning – including reasoning by mathematical induction – is essentially the same as all other thought and, so, at bottom, purely logical in character. As Frege put it:

> Thought is in essentials the same everywhere: it is not true that there are different kinds of laws of thought to suit the different kinds of objects thought about . . . .
>
> The present work will make it clear that even an inference like that from n to $n + 1$, which on the face of it is peculiar to mathematics, is based on the general laws of logic, and that there is no need of special laws for aggregative thought. (Frege 1884, pp. III–IV)

Logicism thus seeks to do away with 'local' differences in reasoning. According to it, such differences are superficial, and disappear once one penetrates sufficiently deeply into the basic nature of the thought in question. Thus, at the epistemologically most important and revealing levels of depth, what is remarkable about mathematics is its homogeneity with the rest of rational thought – not the 'local' character of its forms of reasoning.

This theme of homogeneity also figures prominently in the thought of Frege's forerunner, Leibniz, who took the view that mathematical truths are nothing but disguised forms of logical identities, and that their proofs should therefore be nothing but the analytical 'unwindings' (by application of definitions) of concepts which trace the theorem proved back to the elemental and paradigmatic cases of conceptual containment – namely, analytical truths of the form 'A is A'.

Frege refined this viewpoint both by enriching the assumed logical basis and by restricting its scope. He divided mathematics into two parts: a 'science of number', or 'arithmetic', on the one hand, and a science of spatial intuition, or geometry, on the other; and he distinguished both of these from empirical science. His logicist claim was then focused exclusively on the arithmetic part. In the words of paragraph 14 of the *Grundlagen:*

> Empirical laws hold good of what is physically and psychologically actual, the truths of geometry govern all that is spatially intuitable, whether actual or product of our fancy. The wildest visions of delirium, the boldest inventions of legend and poetry, where animals speak and stars stand still, where men are turned into stone and trees turn into men, where the drowning haul themselves out of swamps by their own topknots – all these remain so long as they remain intuitable, still subject to the axioms of geometry. Conceptual thought alone can after a fashion shake off this yoke, when it assumes, say, a space of four dimensions or positive curvature .... For purposes of conceptual thought we can always assume the contrary of some one or other of the geometrical axioms without involving ourselves in any self-contradiction when we proceed to our deductions, despite the conflict between our assumptions and our intuition. The fact that this is possible shows that the axioms of geometry are independent of one another and of the primitive laws of logic, and consequently are synthetic. Can the same be said of the fundamental propositions of the science of number? Here we have only to try to deny any one of them, and complete confusion ensues. Even to think at all seems no longer possible. The basis of arithmetic lies deeper, it seems, than that of any of the empirical sciences, and even that of geometry. The truths of arithmetic govern all that is numerable. That is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought? (Frege 1884, pp. 20–21)[15]

Frege's view of the basic laws of arithmetic as being logical in character is thus made plain by his insistence that denial of an arithmetic law results in a global failure of rational thought.

Such a view, however, not only asserts a basic homogeneity of arithmetic thought with the other areas of rational thought, but also requires a homogeneity of the laws of logic with each other! For it demands that there be no subsystem of the basic laws that is both conceptually independent of the remaining laws and also strong enough to deserve to be called a body of rational thought. Thus, execution of Frege's logicist program would clearly require more than what logicism is commonly understood to require: namely, the location of a set of principles *basic* enough to be regarded as laws of rational thought and *powerful* enough to imply the laws of arithmetic. It would require as well either that there be no substantial independence or 'separability' among the basic logical laws, or that substantially all of them are required for the derivation of each and every law of arithmetic. Otherwise, there would be a threat of heterogeneity within the basic laws themselves. I am not sure that Frege saw this as clearly as he should have (although it may be what he had in mind in his otherwise curious insistence that the basic laws of thought be kept to a 'small' number (cf. Frege 1884, section 90; Frege 1967, p. 2)).[16] But regardless of whether Frege saw it, it is surely a serious concern for the logicist.

The basic laws of thought identified by logicism, then, must all be of such a character as to qualify them as *fundamental* principles of rational thought. And it is precisely this which produces the basic conflict between Poincaré and the logicists.[17] For, as stated above, Poincaré's view was that mathematical thought possesses its own distinctive principles of reasoning which are *not* fundamental principles of rational thought *per se* but, rather, principles of a more localized logic.[18] Furthermore, he believed that knowledge of such locally distinctive patterns of reasoning is essential to genuine mathematical knowledge, while the logicists believed that the deepest, most genuinely mathematical knowledge is that which consists in a grasp of the reduction of mathematical truths to their logical roots and, hence, to principles whose very essence is their *global* validity.[19] Thus, there is a deep and profound conflict between the epistemological ideals of Poincaré and those of the logicists. As suggested earlier, however, this conflict centers not on the question of whether foundations are to be psychologically realistic but, rather, on the question of whether rational thought is homogeneous (whether, that is, sufficient reason in mathematics comes in one or many different varieties).

It is important for our purposes, however, that Poincaré's basic disagreement with logicism not be confused with that which typifies much of the more recent literature on the subject and takes as central the question whether the *axioms* of the proposed logicist schemes (in particular, axioms which postulate the existence of things – for example, infinite sets) are truly logical in character. For, as was noted in the previous section, Poincaré expressly disclaimed this as his main concern (cf. Poincaré 1902, pp. 31–32), and focused on the methods of inference rather than on the axioms.[20] Moreover, even there his point was not what one might expect – namely, that the inferences used are not purely logical in character. Rather, it was quite the opposite; namely, that they *are* purely logical, and for that very reason cannot be used to produce conclusions that constitute an extension of the mathematical knowledge represented by the premises.

> Verification [which is Poincaré's word for logical proof] differs from true demonstration precisely because it is purely analytic and because it is sterile. It is sterile because the conclusion is nothing but the premises translated into another language. On the contrary, true demonstration is fruitful because the conclusion here is in a sense more general than the premises. (Poincaré 1902, p. 33; brackets mine)

This may make it sound as if the point is this: a *logical transform* $\lambda(S)$ of a sentence S is just like a *definitional transform* of S. Definitional transforms are, of course, *conservative* in a strong sense; that is, the application to S of a definition δ of some expression contained in S can only produce a sentence $\delta(S)$ that expresses the same proposition as that expressed by S. Hence, if logical inference is essentially definitional transformation (that is, if the conclusion of an analytic inference is nothing but the "premises translated into another language"), then the application to S of a logical form of inference λ must result in the production of a sentence $\lambda(S)$ which expresses the same proposition as that expressed by S. This being so, logical inference would not be capable of extending knowledge since it could not produce a conclusion expressing a different proposition from that expressed by the premise(s).[21]

Such an argument might apply to a Leibnizian form of logicism which holds that the truths of mathematics are derivable from logical identities sheerly by application of the appropriate definitions. It would not, however, count against the types of logicism that Frege and Russell advocated, since they explicitly allowed for forms of inference whose conclusions are (or express) different propositions from their premises, and Russell even insisted on the synthetic character of logic.[22] Moreover – and this is the important point – Poincaré's aim was not to establish that logical inference is incapable of yielding any epistemic extension of knowledge whatsoever. Rather, it was that it could not be expected to yield an extension of genuine *mathematical* knowledge.

Thus, the epistemic impotence or non-productivity that Poincaré ascribed to logical inference was specific rather than general, applying only to its use as a means of extending mathematical knowledge. Like Kant before him, then, Poincaré, too, was free to admit that the conclusions of logical inferences might be buried so deeply in their premises as to make them candidates for new knowledge (of a non-mathematical variety) when retrieved through analysis.[23] Thus, to repeat, his point was not that logical inference is epistemically fruitless *simpliciter*, but rather only that it is fruitless as a means of extending mathematical knowledge.

Hopefully, these remarks will help guard against misunderstanding by distinguishing the Poincaréan objection to logical inference from others with which it might be confused. These possible confusions having now been noted, let us turn to the more positive task of developing a few of the key ideas of this Poincaréan viewpoint.

Its cornerstone is the anti-logicist doctrine mentioned earlier; namely, the division of rational thought into irreducibly *heterogeneous* local domains, each with its own distinctive 'logic', if you will. This heterogeneity occurs, moreover, not only at the level of the axioms or first truths of mathematical thought, but also at the level of inference. Thus, Poincaréan heterogeneity is more thoroughgoing than that which would arise solely from those 'pockets' of intuition used to secure knowledge of the first truths or axioms of a body of mathematical thought. It reflects as well the use of intuition in inference, the effect of which is to induce a distinctive logic on a given local domain of thought, thus permitting – indeed obliging – the reasoner to proceed by means other than globally valid steps of inference. Thus, we are reminded once again that Poincaré was not so much opposed to logicism as to the logicization of mathematical proof. For even if the logical status of the logicist's axioms were to be established, this would not provide a way around Poincaré's objection to the use of logical inference in mathematical proof.[24]

But, if rational thought is thus heterogeneous, what is it that creates this local orientation? One thing Poincaré suggests is the need we have for *epistemic condensers* – that is, devices which serve to "abridge our reasonings and our calculations" (cf. Poincaré 1908, p. 440) by packing a whole series of what would be *logical* inferences into the space of a single (non-logical) inference and thus relieving us of the burden of having our knowledge depend on the completion of the cumbersome unabridged logical reasoning. Such condensation signifies – and may even make possible – a grasp of the 'architecture' of a subject, something which Poincaré regarded as an absolutely indispensable ingredient of any truly scientific understanding of a given domain of inquiry. He introduced this notion by means of analogy in the following way:

> Our body is formed of cells, and the cells of atoms; are these cells and these atoms then all the reality of the human body? The way these cells are arranged, whence results the unity of the individual, is it not also a reality and much more interesting?
>
> A naturalist who never had studied the elephant except in a microscope, would he think he knew the animal adequately? It is the same in mathematics. When the logician shall have broken up each demonstration into a multitude of elementary operations, all correct, he still will not possess the whole reality; this I know not what which makes the unity of the demonstration will completely escape him.
>
> In the edifices built up by our masters, of what use to admire the work of the mason if we cannot comprehend the plan of the architect? Now pure logic cannot give us this appreciation of the total effect; this we must ask of intuition. (Poincaré 1908, p. 436)[25]

The epistemic condensation represented by a given non-logical local inference thus signifies not so much a gain in efficiency (though that, too, may be part of its importance) as a grasp of the contribution that the result thus inferred makes to some larger enterprise to which the given local domain of inquiry belongs. Or, perhaps better, it signifies a grasp of how the movement from premises to conclusion contributes to the 'development' of some architectural theme of the local subject-matter. In short, it marks the presence of a comprehending 'universal' (viz., what we are calling a plan or architectural theme of the local domain in question) in the 'differences' (viz., the states of its development signified by the premises and the conclusion, respectively) through which it persists.[26]

Seen in this way, a mathematical inference **I** is composed of three elements: (i) a universal, $U_{I,T}$, which expresses an architectural theme of the local theory **T** to which **I** belongs, and which serves as the modulus of comparison for the premises and the conclusion of **I**; (ii) a premise-set, **p**, which marks a certain 'position' or 'stage' $\sigma_{U(p)}$ in the development of $U_{I,T}$; and (iii) a conclusion, **c**, which marks a 'subsequent' position or stage of development $\sigma_{U(c)}$ of $U_{I,T}$. Consequently, two inferences **I** and **I**′, expressed in the terms of two theories **T** and **T**′, are the same only when $U_{I,T} = U_{I',T'}$, $\sigma_{U(p)} = \sigma_{U(p')}$, and $\sigma_{U(c)} = \sigma_{U(c')}$.[27] And mathematical inferences will typically be distinct from logical inferences, since they will be based on different kinds of universals. Thus, a theory of *mathematical* inference, should one exist,[28] would consist in a specification of the various universals or architectures that serve to organize or unify mathematical thinking and an account of the ways in which the universals in this restricted class affect the differences through which they persist. Likewise, a theory of *logical* inference (assuming that there would be a point to allowing for such a category of inferences) would consist in a specification of a class of cognitive architectures that pertain to topic-neutral reasoning, together with an account of how they affect the differences through which they persist.[29]

Theories of mathematical and logical inference would thus stand side-by-side, rather than in some vertical relationship (signifying subsumption of the former by the latter) to one another. And, it is unlikely that there would be anything like a *general* theory of inference, since it is doubtful that there is sufficient in common among *all* effects of *all* universals on the differences through which they persist to give rise to anything rich and interesting enough to be called a general theory of inference.[30] This is, of course, just a way of articulating the Kantian, anti-logicist theme of Poincaréan epistemology sounded earlier; namely, that rational thought is essentially heterogeneous rather than homogeneous.

The difference between mathematical and logical inference, on this view, is thus one that centers on the choice of universals or architectures under which the premises and the conclusion are to be united: a mathematical inference being one which unites premises and conclusion as a 'development' under a *mathematical* architecture or theme, and a logical inference being one which unites premises and conclusion as a 'development' under a *logical* architecture (if such there be). Our earlier use of the metaphor of 'size' is therefore to some extent misleading. The difference between logical and mathematical inference is not essentially one of 'size' but, rather, of over-arching architecture. Size enters only because the desired topic-neutrality of logical inference forces it to forego appeal to any topic-specific architecture and thus reduces its size to one that is dictated not by a topical architecture but, rather, by a semantical criterion used to individuate propositions generally (since, without a topic-specific architecture to appeal to, all that is left to 'mark' inferential movement is *semantical* change).

The essential distinction between mathematical and logical inferences is thus not well got at by appealing, as Poincaré himself did, to a supposed distinction between analytic and synthetic inference which treats the former as inference in which the conclusion is 'contained in' the premises, and the latter as inference in which the conclusion 'goes beyond' the premises. On the view sketched above, all inference is synthetic, since it all involves a 'putting together' of the premises and conclusion of an inference in such a way as to see the movement from the former to the latter as a 'development' of an architecture.[31] Likewise, all inference is analytic since it involves seeing how the conclusion can be extracted from the premises as a 'development' of the architecture in which they are embedded. Thus, a conclusion's constituting a development of an architecture with respect to a set of premises is sufficient to make it both go beyond and be contained in them.

Hopefully, this account of inference helps clarify the basis for the principle of epistemic conservation which, as noted at the outset of this paper, is the chief structural element of Poincaré's epistemology of proof. According to that principle, an inference from **p** to **q** cannot be used to extend mathematical knowledge from **p** to **q** unless it itself constitutes mathematical knowledge. The basis for that principle, on the present view, is the fact that before an inference from **p** to **q** can be counted as truly mathematical, it must first be fitted into a mathematical architecture. Knowledge of this architecture is the quintessential form of mathematical knowledge. Hence, it is clear both that and why the principle of epistemic conservation holds under the conception of inference sketched here. And, because of that, it is also clear that it is equipped to accommodate Poincaré's basic 'data' for mathematical epistemology: namely, the seeming differences between the epistemic conditions of the logician and the mathematician, and the relative scarcity of mathematical as opposed to logical expertise.

So far, we have only described the 'universals' or 'architectures', which are the key elements of the Poincaréan conception of proof, in terms of their epistemic function or role, which is that of dividing up mathematical thinking into stages or positions and thence to (partially) order those stages into something that can be regarded as a 'development' (i.e., a potential continuation of thought that presupposes a goal and a notion of what it is for that goal to be approached). But, what is it to have knowledge of such an ordering of a potential body of mathematical thought? This is a difficult general question, and one that we can only partially answer here. However, one thing seems sure, and that is that the knowledge in question is not to be seen as consisting merely in a knowledge of the specific proofs and theorems that would constitute a 'development' of the area of thought in question. Grasping an 'architecture' is not a matter of seeing the specific results subsumed under it.

Does it, then, consist simply in an ability to tell of a given proof, when presented with it, whether or not it contributes to the development of the field in question? Seemingly not, since having a grasp of an architecture is supposed to function not only to control the mathematician's reactions to possible proofs of which she may become aware, but also to *guide* her in the *discovery of* proofs which contribute to the development of the field. Hence, Poincaréan architectures are to serve not only as standards with which our epistemic choices are to accord, but also as guides which direct those choices.[32]

Knowledge of an architecture is thus something like knowledge of a strategy for playing a game, though not knowledge which presupposes a *complete* knowledge of how to get from one's present circumstances to the goal. It thus presupposes (i) the having of a goal, and the ability both to (ii) determine of a given course of proof-activity whether or not it is in conformity with that goal, and (iii) to discover particular proof-activities that are in conformity with it.

Thus, mathematics may be seen as a combination of special goals and special strategies for attaining them. Still, it is no game. In part this is no doubt due to the 'seriousness' of its overall goal as a science – namely, the development of our epistemic holdings. But, even more, it may be due to the fact that the strategies of mathematics have a sort of *regulative* status that strategies in games do not generally have. In a game, it is not strategies but *rules* and *goals* that have regulative status. I cannot be said to be playing chess if I move my bishops in straight lines, or if I adopt as my goal the mating of my own king. I can, however, play it without having a good strategy for mating my opponent's king. Playing the game *well* requires the employment of a good strategy; playing the game *at all* does not. To play the game at all requires only that I operate within the boundaries of the rules and that I make an appropriate choice of goals; it does not require that I make use of a good strategy.

On the Poincaréan view of mathematics, however, things are different. There, the strategies do have regulative force. They *are* rules. Not to employ a mathematical strategy (i.e., a strategy which embodies some genuine mathematical – as opposed to general, multi-purpose – insight) is simply to fail to do mathematics at all. One can, of course, do mathematics less well than it might be done – and also less well than others actually do do it. One can, perhaps, even do it poorly – though that's a trickier matter.[33] But this only means that not all genuinely mathematical strategies are on a par with each other as means of pursuing a given mathematical goal, and that some of them might even be downright bad.[34] It does not controvert the basic claim that, in mathematics, strategies have regulative status.

None of this should, of course, be taken as implying that without a grasp of a mathematical architecture there can be no inferential extension of *any* kind of knowledge (even knowledge of mathematical propositions). That would be both rash and incorrect. Rather, what we have been arguing is that without a grasp of a mathematical architecture there can be no inferential extension of mathematical knowledge. Inference, as we have presented it, is fundamentally a case of similarity through change. Moving from premises to conclusion brings a change; however, if the inference is to be valid, the premises must somehow be 'reflected' in the conclusion and thus persist *through* the transition to the conclusion. Such binding of premises to conclusion is basic to anything that truly is to count as inference; and it is the particular 'way' or 'mode' in which the premises are reflected in the conclusion (e.g., logically, mathematically, etc.) that determines what *kind* of inference an inference is.

In the Poincaréan epistemology that we have been sketching here, the similarity through change of truly *mathematical* inference is to be accounted for by subsumption of the premises and conclusion under a common *mathematical* architecture. One thus finds the premises reflected in the conclusion in a *mathematical* way. Subsumption under other kinds of architectures would give rise to different kinds of reflection. Architectures thus function as universals, binding the premises and conclusion of an inference together into the sort of 'unit' that is necessary to make it an *inference* rather than a mere *sequence* of propositions or judgments. Knowledge of architectural binding is, consequently, an essential part of that which is required for extension of a given *kind* or *type* of knowledge by means of inference. Knowledge of binding by a mathematical architecture – rather than such things as preservation of certainty and/or *a priority* – is thus the crucial difference separating mathematical from non-mathematical inference, and is the distinctive feature of mathematical proof.

4. THE NATURE OF MATHEMATICAL RIGOR

The view of proof sketched in the last section brings with it the need to develop an accompanying conception of rigor since the standard *logical* conception of rigor can no longer be applied. On the Poincaréan account, mathematical proof no longer proceeds in logic-sized steps but, rather, in steps determined by the 'metric' of a given mathematical architecture. The inferences in a proof themselves comprise substantive pieces of mathematical insight, and this is something that runs counter to the very core of the modern, logical conception of rigor.

In conceiving of a mathematical inference as the joint participation of its premises and conclusion in a distinctively *mathematical* universal, the Poincaréan conception of proof builds non-logicality into the very essence of mathematical inference. It regards the 'filtration' of topic-specific knowledge from mathematical inference as tantamount to destroying it. Consequently, it stands squarely opposed to the modern, logical account of how rigor in proof is to be achieved, and it must therefore provide an alternative account.

This it does by offering a different *conception* of rigor altogether. On this new conception, the basic ideal is the same: namely, to eliminate 'gaps' in mathematical reasoning. However, both the conception of what constitutes a 'gap' and, consequently, the prescribed method for achieving their elimination are different. A 'gap' is no longer a *logical* gap but, rather, a gap in *mathematical understanding*. Gaps of these two types are by no means equivalent, since it is possible both for there to be a logical gap where there is no gap in mathematical understanding and for there to be a logically gapless sequence of propositions that is nonetheless not bound together by any mathematical understanding. The elimination of gaps thus no longer calls for the *exclusion* of topic-specific information in an inference (which is what logical gaplessness demands) but, rather, for the *inclusion* of a mathematical universal to fill what would otherwise be a mathematical gap between the premises and the conclusion.

But, is it then not clear that the Poincaréan criterion of rigor fails to guarantee a complete and fully explicit identification of those propositions upon which the conclusion of a proof rests? And, if this is so, is it not also clear that there is a serious deficiency in the Poincaréan conception of rigor, since the whole point of pursuing proof rigorously is to gain a clear understanding of that upon which justified belief in a given mathematical proposition can be made to rest?

It is, we believe, possible to answer both questions in the negative. To see that this is so, however, it is important to distinguish the notion of a proposition's *mathematically* resting on another from that of a proposition's *logically* resting on another. Propositions that belong to the logical basis of a proposition need not belong to its mathematical basis. Thus, if P mathematically guarantees C, then, whether or not P logically guarantees C, it (i.e., P) constitutes a complete set of propositions upon which C mathematically rests.[35] Hence, in specifying it, one need leave no propositions out of the relevant basis for C; and this means that it is possible to obtain a full account of those propositions upon which a given mathematical belief rests, without thereby obtaining a logically complete basis for it.

The above remarks should not, however, be taken as suggesting that logical rigor has no role whatsoever to play in the improvement of mathematical knowledge. Indeed, we believe that it does. Specifically, in times of epistemic crisis, when it becomes necessary to revise epistemic holdings which, judged from a purely mathematical point of view, seem unimpeachable, generating logically complete bases may prove to be the only, or at least the optimal, way of proceeding. Such a procedure has the advantage of explicitly exposing certain assumptions which mathematical rigor does not bring to light. And, in so doing, it may turn up something implausible and, hence, something defeasible, in our tacit assumptions. But whether or not this is the result, the generation of logically gapless bases for theorems will tend to enlarge the range of potential candidates for revision and, other things being equal, this is a virtue in situations where revision is demanded. It is important to realize, however, that even here it is logical rigor that serves the interests of mathematical rigor and not the other way around. For the point of such 'logicization' is not to replace mathematical proof but, rather, to correct it – not to abandon the use of mathematical universals but, rather, to perfect it.

5. CONCLUDING REMARKS

Let us close with a few remarks intended to guard against misunderstanding and, at the same time, to deepen certain parts of our analysis. Central to our concerns here is a potential misunderstanding which arises from a natural response to the epistemology of proof sketched above. The basic idea behind this response is that it should be possible somehow to 'express' or 'encode' every universal or architecture appealed to in the course of a Poincaréan proof as an axiom and, that done, to make every inference into a purely logical inference. The process for carrying this out consists of the following two steps: (i) express each Poincaréan inference as an axiom in conditional form (the antecedent being the premise, and the consequent the conclusion of the inference being codified); and (ii) replace that inference by an instance of modus ponens. In this way, the reasoning goes, every Poincaréan proof can be transformed, without epistemic loss, into a purely 'logical' proof (i.e., a proof whose *inferences* are purely logical).

We believe that this argument turns on a failure to recognize a subtle, yet crucially important, difference between the Poincaréan proof and (what we might call) its *modus ponens counterpart*; a difference which can only be got at by probing more deeply into the epistemic mechanisms underlying the two. Let us turn, then, to a more careful examination of each, so that we may more fully appreciate the differences between the grounds which each provides for its conclusion.

In the case of a Poincaréan proof, the key feature is the grasping or intuiting of a mathematical architecture between **p** and **c**. **p** is seen to bear an architectural connection to **c**, and it is the grasping of this architecture – and seeing that it joins **c** to **p** – that is crucial to the Poincaréan inference from **p** to **c**. This is different from simply recognizing the *warrant for* connecting **p** and **c**, which is supplied by a grasp of the subsuming architecture. Grasping an architecture linking **p** and **c** surely provides one with a warrant for connecting **p** and **c**. And this warrant surely makes acceptable both the inference from **p** to **c** and the conditional proposition 'if **p**, then **c**'. Grasping an architecture is not, however, epistemically reducible to the provision of such warrant; it does more than simply justify (or justify with certitude, or justify with certitude *a priori*, etc.) the belief that if **p**, then **c**. It reveals **p** and **c** as *mathematically* connected.

The epistemology of (what we are calling) logical proof is entirely different. Logical relations between propositions, at least on the classical view, are essentially relations between their truth-values. Likewise, the essential function of a warrant is to establish what the truth-value a given proposition is. Thus, with the truth-value of a given proposition **p** having been established, the recognition of a logical relationship between **p** and another proposition **c** essentially amounts to one's having the ability to establish the truth-value of **c**. Thus, logical inference is, epistemologically speaking, a means of extending the ability to determine truth-value from one proposition to another.[36]

This is essentially the epistemology of the modus ponens counterpart of a Poincaréan proof. It shows us why, even if knowledge of the premise 'if **p**, then **c**' is *based on* a grasp of the mathematical architecture linking **p** to **c**, the Poincaréan proof and its modus ponens counterpart are essentially different. For even though the warrant for 'if **p**, then **c**' may be based on a grasp of a mathematical architecture linking **c** with **p**, the modus ponens counterpart nonetheless *abstracts away from* this grasp of architecture itself and focuses instead on its net classical effect; that is, on the classical semantico-epistemic status (e.g., truth, certitude of truth, *a priori* certitude of truth, etc.) and logical form of the belief (viz., 'if **p** then **c**') which it warrants. It seeks to replace grasp of a mathematical architecture connecting **p** and **c** with *recognition of* its classical effects (i.e., the semantico-epistemic status and logical form of 'if **p** then **c**'). It treats grasp of an architecture as merely the means by which the classical semantico-epistemic status of 'if **p**, then **c**' is established while viewing the semantico-epistemic status itself, rather than the means of establishing it, as the matter of primary epistemic importance.

This leads us to certain observations concerning the nature of what we are here referring to as 'logical proof'. The first is that, by an act of abstraction, it separates or detaches the net classical effect of a warrant or justification from that warrant or justification itself. It then replaces the actual grasp or comprehension of the underlying justification with *reflection on* the semantic or epistemic status thus abstracted from it. In addition to this, the 'net effects' with which it is concerned (e.g., truth, warrantedness, unwarrantedness, warrantedness to degree **n**, warrantedness with certainty, warrantedness *a priori*, and so on) show too little sensitivity to the particular characteristics of the warranting processes producing them to be expected to distinguish a mathematical from a non-mathematical grasp of a connection between propositions in the Poincaréan sense. The 'logician' who would replace a Poincaréan proof with its modus ponens counterpart can have all the certitude or *a priori* certitude for his beliefs that Poincaré's mathematician does. Still, though he may enjoy the net classical effects of architectural grasp, and though those effects may be abstracted from that grasp, it is not that grasp itself (but, rather, a reflection on the *logical* connections between the propositions making up his inference, and how these connections provide for the transfer of the requisite classical semantico-epistemic attributes between these propositions) which guides his inference. Indeed, it is precisely this which is omitted through the core act of abstraction which gives rise to a 'logical' proof.

It is, of course, entirely natural that logic should abstract away from and, hence, be insensitive to, all but a very few characteristics of warrants. Otherwise, it could not achieve that universal scope of applicability which, by its very nature, logic is supposed to have. Thus, abstraction away from all but such relatively coarse epistemic features as truth and degree of certainty is programmed into logic by the very nature of its goals. As we hope the above remarks suggest, this programmed insensitivity is the point of departure of the Poincaréan's critique of attempts to 'logicize' genuinely mathematical proofs. Grasp of a mathematical universal linking **p** and **c** is not epistemically reducible to – and, hence, not epistemically replaceable by – reflection on its classical effects. This, as we see it, is the essential point behind the *principle of epistemic conservation*, which was identified earlier as a key feature of the Poincaréan conception of proof.

Thus, we see that the central epistemological categories of the Poincaréan conception of proof are to a significant extent different from those of the traditional 'logical' conception. It would, of course, be nice to provide a more illuminating description of those categories than that which is captured in such phrases as "grasping a mathematical universal" and "mathematical intuition". That, however, may not be possible since such notions may simply form the primitives from which a Poincaréan mathematical epistemology must proceed. But primitive or not, such categories appear to be necessary if, as Poincaréan epistemology would have it, we are to be able to account for the seeming differences separating the epistemic condition of the 'logician' from that of the genuine mathematician.

NOTES

* The author would like to thank the Alexander von Humboldt Foundation for its generous financial support. Thanks also to Dick Foley for useful comments on an earlier draft.

[1] Cf. Poincaré (1902, 1905, 1906, 1908, 1913), and Russell (1905, 1906). But though, historically speaking, Poincaré's debate was carried on with Russell, his ideas find a more striking and worthy opposite in the writings of Frege, whose philosophical defense of logicism I regard as superior to that of Russell's. For this reason, I will give Frege's ideas a prevalence that the reader might otherwise regard as surprising given Poincaré's seeming ignorance of them.

[2] This is a theme that is sounded repeatedly in Poincaré's writings. Cf. 1905, ch. 1 (in particular, pp. 13–26, 29–31, and 36–37); 1908, Intro., ch. 3, bk. I; chs. 2, 3, bk. II. Throughout this essay, English translations (along with accompanying page references) of passages from 1902, 1905, and 1908 are from the authorized translations of those works collected in Halsted. À propos 1913, the translation and pagination are from the translation by Bolduc.

[3] Cf. Poincaré 1908, ch. 2, bk. II (p. 438, Halsted). These ideas of Poincaré's resonate with some of Brouwer's philosophical views. For more on this, see Detlefsen (1990).

[4] Cf. Kitcher and Aspray, pp. 61–81.

[5] Cf. Frege (1884, p. 23) where he cites the views of Leibniz with approval. Leibniz (cf. Bk. IV, ch. vii, par. 9) maintained that there is a "natural order of truths" that is not to be confused with the order of discovery or awareness or belief. Rather, the ordering of this "natural order" is an objective, metaphysical ordering wherein a given known truth is not only the basis of our judging another proposition to be true, but also the "cause" of that proposition's truth.

> A reason is a known truth whose connection with some less well-known truth leads us to give our assent to the latter. But it is called a reason especially and *par excellence*, if it is the cause not only of our judgement but also of the truth itself. (Leibniz, bk. IV, ch. xvii, par. 3)

[6] We say this despite the fact that Poincaré himself once made an unfavorable comparison between the views of Hermite and those of the so-called Cantorians (cf. 1913, pp. 72–74) on the grounds that the latter put too much emphasis on "epistemology" and not enough on "psychology". However, the 'epistemology' of these remarks is not the epistemology of the present discussion; nor is the 'psychology' the pschology which was opposed by the logicists. Rather, the 'epistemology' is a radically anti-Kantian affair which denies that "all that we can know of [a reality that is exterior to and independent of us] depends on us" (cf. p. 74; brackets mine). Complementarily, the 'psychology' consists in the recognition of a Kantian mental element (like the categories of thought) in knowledge.

[7] On a simple subjectival typification, one starts with a sorting of propositions according to subject-matter, and then constructs an epistemic typification on that by sorting knowledge according to the subject-matter of its content. Thus, knowledge of a proposition **p** would be classified as belonging to a certain epistemic type just in case **p** were classified as belonging to a certain subject-matter. Applied to the case we're interested in – namely, mathematical knowledge – such a classification scheme would work like this: **p** is

mathematical knowledge just in case (i) **p** is known, and (ii) **p** is a proposition of mathematics.

Modifications of a subjectival typification can be obtained by modifying its non-subjectival clause. Thus, for example, one could obtain a modification of the above subjectivally typified characterization of mathematical knowledge by altering clause (i) to require that **p** be known with certainty, and/or *a priori*. The principle of typification which we will suggest as a basis for Poincaréan epistemology belongs to this general category of modified subjectival typifications. That is, it treats subjectival considerations as necessary but not sufficient conditions for mathematical knowledge. Hence, it demands more of mathematical knowledge than simply that it be knowledge whose content is mathematical. It sees certain features of what might be called *noetic mode* as being as important a part of mathematical knowledge as content.

Its chosen variant of noetic mode, however, is not merely some combination of the classical ones mentioned above (i.e., certainly, and/or *a priority*). Indeed, it is different in spirit from such traditionally accepted properties of mathematical knowledge. For the traditionally accepted properties apply as well (and perhaps even better!) to logical knowledge as to what is more properly mathematical. Yet Poincaré's conception of noetic mode is intended to exclude logical knowledge. More on this later.

[8] What Poincaré actually says (cf. 1908, ch. 3, bk. I (p. 385, Halsted)) is this: if one can "perceive at a glance the reasoning as a whole", then she "need no longer fear lest [she] forget one of the elements for each of them will take its allotted place in the array" (brackets mine). Given the larger context in which this remark occurs, however, it is not altogether clear whether it is to be taken as stating the view expressed above, or a view according to which eliminating gaps in understanding leads to the elimination of logical gaps. We prefer our reading because it seems to fit better with Poincaré's belief in the essential non-logicality of mathematical reasoning. (Why, if mathematical reasoning is non-logical, should perfecting mathematical understanding lead to the elimination of logical gaps in a proof? And why should this be regarded as something desirable?)

[9] The paragraph referred to is headed by the question "How is Pure Mathematics Possible?". The text is as follows:

> Here is a great and established branch of knowledge, encompassing even now a wonderfully large domain and promising an unlimited extension in the future. Yet it carries with it thoroughly apodeictic certainty, i.e. absolute necessity, which therefore rests upon no empirical grounds. Consequently it is a pure product of reason, and moreover is thoroughly synthetical. [Here the question arises:]
>
> "How then is it possible for human reason to produce a cognition of this nature entirely *a priori*?"

[10] By a Leibnizian form of logicism we mean a strong form of logicism which does not put set-theoretic principles like Frege's or type-theoretic principles like Russell's into the logical foundations. Rather, it claims to use only such immediately logical or analytical principles as the so-called law of identity and the principles of syllogism. Thus, Poincaré's fundamental opposition to logicism was not based on a suspicion of the less clearly analytic principles (e.g., axioms of comprehension, infinity and reducibility) that Frege and/or Russell put into their 'logical' bases. Rather, it rested on what Poincaré regarded

as an unacceptable consequence of its success; namely, that the whole of mathematics would thereby be reduced to a body of 'tautologies'.

[11] Cf. also 1908, chs. III, IV (pp. 452, 462, Halsted).

[12] What is here, by inference, being termed the 'logical' conception of rigor might also be called the 'logicist's' conception of rigor. This is so because it was introduced not for the sake of exposing and eliminating steps in our reasoning for which our mathematical understanding does not provide a warrant but, rather, to force all elements of that understanding out into the open so that their *logicality* might be determined. This is part of the logicist's task, since he must show that every truth of mathematics can be established on purely logical grounds.

[13] To put it another way, Poincaré's primary challenge to Russell and Cantor was not to defend the distinctively logicist idea that the basic laws of mathematics are logical in a character but, rather, to defend the accompanying view (shared by many non-logicists as well) that the logicization of mathematical proof represents an ideal of mathematical rigor and, hence, something to which mathematical practice ought to aspire.

[14] To my knowledge, Poincaré never gave another specific example of a distinctively mathematical form of inference. Still, he believed that there were kinds of intuition other than arithmetical (cf. his remarks on *analysis situs* in 1913, pp. 25ff.), and this suggests that the existence of other kinds of distinctively mathematical inferences would not be foreign to his thought. Indeed, he says (1908, Bk. II, ch. III, sec. III (p. 452, Halsted)) that he did not mean to suggest in his earlier writings that all mathematical reasoning can be reduced to induction, but only that it is the simplest example of the general type of reasoning that he has in mind, and that all other representatives of this kind share the same "essential characteristics". It may, however, be that he believed mathematical induction to enjoy a place of special distinction because of its newly confirmed "centrality" – a centrality made apparent by the impressive efforts of Dedekind and Weierstrass to "arithmetize" analysis. This moved (second-order Peano) arithmetic and, with it, (second-order) mathematical induction much closer to the center of the mathematical arena. Hence, it gave mathematical induction a certain prominence among mathematical forms of inference.

This may have been the reason why Poincaré made mathematical induction the focal case of a distinctively mathematical inference. Still, we should not lose sight of the fact that he also emphasized the fact that there are limitations to how fully the founding ideas and conceptions of topology and analysis can truly be arithmetized (cf. 1913, p. 29).

[15] Though Frege sided with Leibniz on the nature of arithmetic, he sided with Kant on the nature of geometry, saying

> . . . I consider Kant did a great service in drawing the distinction between synthetic and analytic judgements. In calling the truths of geometry synthetic and a priori, he revealed their true nature. And this is still worth repeating, since even today it is often not recognized. If Kant was wrong about arithmetic that does not seriously detract, in my opinion, from the value of his work. His point was that there are such things as synthetic judgements a priori; whether they are to be found in geometry only, or in arithmetic as well, is of less importance. (Frege 1884, pp. 101–02)

[16] Roughly, the idea here is that if the number of basic laws is kept 'small', then there is less chance that any proper subsystem of them would amount to anything substantial

enough to be regarded as an autonomous domain of reasoning. Hence, removal of any one of them would effectively destroy reasoning as we know and think of it, and so *substantially* all of them would have to be involved in the proof of each theorem. Of course, for such argument to work, one has to make various auxiliary assumptions (e.g., that no basic law is used only in the derivation of a relatively small and/or unimportant portion of the entire system of theorems).

Another task that Frege seems not to have seen too clearly is the need to keep the basic laws of thought from being *too powerful*; specifically, from being powerful enough to serve as a basis for geometry as well as arithmetic. For if they were to become this powerful, then Frege would not be able to sustain his agreement with Kant concerning the synthetic *a priori* character of the laws of geometry, since the posited difference in revisability of geometrical and arithmetical laws could not then be accounted for. This may actually be a serious problem for Frege, since the 'science of number' for which he intends to provide a logicist basis is apparently supposed to be strong enough to include such real number theory as would be necessary for an analytic version of geometry. Of course, one must be able to get the apt representation of spatially given continua and transformations thereupon by their analytically described counterparts. However, it is not clear – at least not to me – that this requires spatial intuition rather than what Frege would have regarded as purely 'conceptual' thinking. But whatever the case, it seems clear that Frege owes us some explanation.

[17] Something like this basic point holds for Russellian as well as for Fregean logicism. For Russell, too, intended his basic principles to be ubiquitously manifested features of rational thought, or at least of *mathematical* thought.

[18] Could an emphasis like Poincaré's on the importance of locally distinguishing features of an area of thought perhaps be *combined with* an emphasis like the logicist's on the ultimate reducibility of all local areas of thought to global principles of rational thinking? That is, might it be possible to have *both* a set of locally distinctive derived results *and* a globally valid set of ultimate axioms? The answer, I think, is 'No'. For assuming that the theorems were derivable using only globally valid principles of reasoning, as the logicists insist, they (i.e., the theorems) would be derivable from globally valid axioms using globally valid principles of inference. There is some intuitive force to saying that this would make the theorems globally valid, too. Still, I don't know how to prove such a claim generally, nor even in the case of Poincaréan epistemology. Allow me to explain.

If one assumes a Tarskian notion of consequence, the claim is easy to prove. For then a theorem resulting from the application of globally valid rules of inference to globally valid premises must itself be globally valid (i.e., true in every interpretation of its language).

One can even see how to put together a proof for someone like Frege who had a rather more epistemic conception of consequence. On his conception, a proposition is not to be identified with its set of models, as is the case in Tarskian semantics but, rather, with its set of ideally rational believing agents, where these agents are to be thought of as demarcated by the *a posteriori* and *synthetic a priori* information they have. Hence, a globally valid axiom is one that is believed by all ideally rational agents regardless of the *synthetic a posteriori* information they possess; and a globally valid principle of inference is one such that every ideally rational agent who believes its premise(s) will also believe its conclusion. It follows, therefore, that the conclusion of a globally valid inference with globally valid premises will itself be globally valid (i.e., believed by all the ideally rational agents regardless of their *synthetic a posteriori* information).

Poincaré, too, had an epistemic notion of consequence, but it was radically different from Frege's. His epistemic states (or attitudes) were *typed* so that it is not only sheer preservation of belief per se but also preservation of belief-type that must be considered in connection with inference. Moreover, his epistemic types were more finely demarcated than those (viz., the *a priori* vs. the *a posteriori*) which Frege took over (with modification) from Kant. In particular, he seems to have had a category of distinctively mathematical information marked, at least typically, by its very lack of globality. Because of this, his epistemic conception of consequence would have to be radically different from that of the Fregean. For extending the Fregean conception would mean counting a conclusion as mathematically known if the premises were mathematically known and the inference taking one from the premises to the conclusion were logically valid (i.e., such that every ideally rational agent who believed the premises must also believe the conclusion). This is so because the mathematical believers would form a subset of the class of ideally rational agents and, hence, would form the class of believers of the conclusion, making (on the Fregean conception) the conclusion a mathematical belief . . . contrary to the way Poincaré seemed to think of things. Hence, on Poincaré's scheme it does not seem possible to represent a mathematical belief simply as (a belief shared by) the (or a) class of mathematical believers, as is suggested by the Fregean model. Indeed, it seems as if mathematical knowers might best be represented as believers *lacking* certain beliefs (e.g., those not marked by a distinctively local validity). If this is so, then the correctness – nay, even the coherence – of the idea that the application of globally valid rules of inference to globally valid premises guarantees globally valid conclusions is unclear. For there is seemingly no convincing way to argue for a connection between non-belief in P and non-belief in I(P, Q) (i.e., an inference from P to Q), on the one hand, and non-belief in Q, on the other. Hence, as was remarked earlier, there is no way to show in general that the application of globally valid principles of inference to globally valid premises results in globally valid conclusions. Specifically (and ironically!), there is no way of doing so for the Poincaréan viewpoint.

[19] As was noted above, in the view of both Leibniz and Frege there is an objective, metaphysical ordering of the truths of mathematics. According to them, one's epistemic mastery of a subject is optimal when she grasps the objective relation which orders its truths. Correspondingly, truths are epistemically mastered when their place within the objective hierarchy is determined, and those proofs are epistemically optimal which display that segment of the hierarchy connecting the truth being proved with the foundational – globally valid – truths. Similar views are espoused in Aristotle (cf. Book I, chs. i–x) and Bolzano (cf. paras. 198, 401 and 525).

[20] It should perhaps be noted that Poincaré's opposition to logicism would therefore extend even to such neologicistic programs as might give up claiming analyticity for the axioms, but still obliterate the 'local' structure of mathematical reasoning.

[21] When more than one premise is involved, I am assuming that the point would be this: the concluding sentence produced by applying an analytic form of inference would have to be equivalent either to one of the premises or to some conjunction of them.

[22] When we speak of a conclusion of an inference expressing a different proposition from its premises, what we mean is that it does not express the same proposition as is expressed by any conjunction of its premises.

[23] Cf. secs. 36–37 of Kant's *Logic*.

[24] The logicist is, of course, also committed to the use of logical inference in mathematical proof. Nor is this only a reflection of his general belief in the homogeneity of rational thought, for he also takes it as necessary for attaining the kind of rigor he requires of his proofs. His proofs must be conducted in such a way as to eliminate the threat of non-logical assumptions surreptitiously entering his inferences. Hence, his inferences must themselves be of a logical character.
[25] Cf. Poincaré (1905, ch. I (pp. 217–18, Halsted)) for a similar point.
[26] How about an example of an 'architecture'? The only one that Poincaré himself provided is that concerning the natural numbers. Grasping the architecture of the natural numbers supplies one with that form of inference which Poincaré regarded as mathematical reasoning *par excellence* – namely, mathematical induction. However, this kind of architecture (which may be thought of as the architecture of a domain of *objects*) is at least conceptually distinguishable from an architecture of a field of thought (which is a family of *results*, not objects). Poincaré did not explicitly distinguish these two kinds of architectures; and this despite the fact that much of what he said seems to depend on there being such a distinction. Perhaps this was because the two types of architectures are clearly related, and that it is architectures of results that are of ultimate concern to an epistemology such as his. Architectures for domains of objects will typically induce architectures (or perhaps sub-architectures) on results; a fact that is made clear by inferential ordering of propositions effected by mathematical induction – an inferential ordering of propositions that is only made possible by the inductive ordering of the natural numbers. Whether all inferential orderings are ultimately based on some feature of an object ordering is a difficult question, and one that we do not know how to answer. But it seems clear that we would have no reason to pay less attention to architectures on domains of results that might arise from other sources.
[27] This conception of inference treats the universal $U_{I,T}$ as at least inducing (if not being literally identical with) a set Σ of 'stages' or 'positions' of its realization, together with an ordering (or partial ordering) O defined on Σ. Thus, when the two universals $U(= U_{I,T})$ and $U'(= U_{I',T'})$ are identified, this means (at least) that $\Sigma = \Sigma'$ and $O = O'$.
[28] And, it is by no means clear that it does, since that depends on whether there is some essential feature common to all mathematical architectures. It might be instead that the various branches of mathematics each have a distinctive architecture of their own, and that each therefore gives rise to its own theory of inference. This is in keeping with what Poincaré has to say about mathematical reasoning since his point is simply that the mathematician always makes use of *some* local architectural knowledge or 'intuition' in her inferences, not that she always makes use of the *same* one. On this view, one would have to rely on something like a notion of 'family resemblance' to account for any commonality that might be thought to bind mathematical reasoning together as a whole, since one seems to be dealing with a similarity that is not metrizable (on pain of collapsing the local distinctions in question).
[29] It may be that a logical architecture is distinguishable primarily by what it lacks rather than what it has. That is, its only distinguishing features may be its preservation of such relatively coarse epistemic features as *a priority* and/or approximate degree of certainty, and its attempt to base a metric for epistemic 'development' or 'advance' on a *semantical* criterion for individuating propositions.
[30] The only thing like a general theory of inference would be the general account of the basic elements or structure of inference sketched above. But that is not what we're thinking of as a general *theory* of inference. The general account of the structure of

inference only says that every inference makes use of an architecture, and a way of parsing and ordering that architecture into stages of 'development'. But it does not presume any significant commonality either among the architectures or the scheme of stages associated with each. What we are referring to as a general *theory* of inference would do precisely that.

[31] This is in addition to the fact that all inference is also synthetic in the sense that it involves uniting (possibly) several premises into a single 'message' (the conclusion).

[32] If the ability in question is understood in such a way as to signify infallibility, the condition would also be too strong. In mathematics, as elsewhere, development can include some false starts. Thus, allowance must be made for a kind of grasp of architecture which does not rule out the possibility of false starts, but which nonetheless gives one *some* guidance. Architectural grasp thus appears to admit of degrees. Hence, in the final analysis, Poincaréan epistemology may also produce a conception of mathematical knowledge which admits of degrees.

[33] It is trickier because at some point it becomes hard to see the difference between doing mathematics poorly and not doing it at all. Just as, I suppose, there comes a point where it is hard to make out a difference between one who plays chess very poorly and one who doesn't really play it at all. The clearest cases of not playing chess are probably where either the rules for moving the pieces are broken, or where one does not play according to the goals of the game (e.g., where one would seek to put himself into checkmate rather than his opponent). But one can approach, in effect, the latter of these paradigmatic failures simply by being a consummately poor strategist. There comes a point where one's strategy is so bad that it is *as if* her intention were to mate herself. (There would, of course, be a difference in the intentions of the poor strategist and the chooser of false goals. However, that may only tell us that there is a difference between actually playing chess and sincerely intending to.) Perhaps the same is true of mathematics (as conceived by the Poincaréan) – there may come a point where one's choice of strategies is so poor that there is no meaningful difference between doing mathematics poorly and not doing it at all.

[34] Actually, it doesn't even mean that. For failure to do as well as might have been done might be due to poor execution of the strategy selected rather than poor choice of strategy.

[35] There may, of course, be no such thing as *the* set of propositions upon which C rests. The same, however, is true of the axiomatic approach, since it does not guarantee existence of a single, unique proof of any theorem and, indeed, in some cases (e.g., cases where the theory is not finitely axiomatizable, where not every axiom is used in every proof, or where the axioms of the theory are not independent), a *plurality* of proofs may be necessitated. We should also like to point out that though we speak here as if inferences are composed of propositions, we are not officially committing ourselves to such a view. What we have to say could be made to fit as well with conceptions of inference which see it as composed of beliefs or judgments. Such an alteration would only force us to speak of the relationship between the contents of the various beliefs or judgments making up an inference.

[36] Modifying this to say that it is a means of extending the ability of one to determine *a priori* truth-value from one proposition to another does not change the view for our purposes. For just as the Poincaréan would insist that there is more to mathematical knowledge that merely determining the truth-value of a mathematical proposition, so, too, would she insist that there is more to mathematical knowledge than, say, the certain

determination, or the *a priori* determination with certitude, of the truth-value of a mathematical proposition. No such modification shows any promise as a means of capturing what Poincaré seems to have had in mind by grasp of an architecture. Thus, viewed from the vantage of the categories of classical mathematical epistemology, Poincaré's notion of grasping an architecture appears to be a primitive notion.

REFERENCES

Aristotle: 1975, *Posterior Analytics*, trans. with notes by Jonathan Barnes, Clarendon Press, Oxford.

Bolzano, B.: 1973, *Theory of Science*, ed. with an introd. by Jan Berg, English trans. by Burnham Terrell, D. Reidel, Dordrecht.

Detlefsen, M.: 1990, 'Brouwerian Intuitionism', *Mind* **99**, 501–34.

Frege, G.: 1884, *Die Grundlagen der Arithmetik*, in *The Foundations of Arithmetic*, trans. by J. L. Austin, Basil Blackwell, Oxford [reprint 1950].

Frege, G.: 1893, *Grundgesetze der Arithmetik*, Vol. I, H. Pohle, Jena.

Frege, G.: 1967, *The Basic Laws of Arithmetic*. English trans. by M. Furth of parts of Frege 1893, University of California Press, Los Angeles.

Goldfarb, W.: 1988, 'Poincaré Against the Logicists', in Kitcher and Aspray, pp. 61–81.

Halsted, G.: 1946, *The Foundations of Science*, authorized trans. of Poincaré's *Science and Hypothesis*, *The Value of Science*, and *Science and Method*; special preface by Poincaré with introd. by Josiah Royce; The Science Press, Lancaster PA.

Kant, I.: 1974, *Logic*, English trans. with introd. by R. Hartman and W. Schwarz, Bobbs-Merrill, Indianapolis.

Kant, I.: 1977, *Prolegomena to any Future Metaphysics*, Carus trans., revised by J. W. Ellington, Hackett, Indianapolis.

Kitcher, P. and W. Aspray (eds.): 1988, *History and Philosophy of Modern Mathematics*, University of Minnesota Press, Minneapolis.

Leibniz, G.: 1982, *New Essays on Human Understanding*, trans. and ed. by P. Remnant and Jonathan Bennett, Cambridge University Press, Cambridge.

Poincaré, H.: 1902, *La Science et l'hypothèse*, Ernest Flammarion, Paris.

Poincaré, H.: 1905, *La valeur de la science*, Ernest Flammarion, Paris.

Poincaré, H.: 1906, 'M. Poincaré's *Science et hypothèse*', *Mind* **15**, 141–43. (This is a reply to Russell 1905.)

Poincaré, H.: 1908, *Science et méthode*, Ernest Flammarion, Paris.

Poincaré, H.: 1913, *Dernières pensées*, Ernest Flammarion, Paris [trans. by J. W. Bolduc as *Mathematics and Science*: *Last Essay*, Dover, New York, 1963].

Russell, B.: 1903, *The Principles of Mathematics*, George Allen and Unwin, London.

Russell, B.: 1905, 'Review of *Science and Hypothesis* by H. Poincaré', *Mind* **14**, 412–18.

Russell, B.: 1906, 'Rejoinder to Poincaré [1906]', *Mind* **15**, 143.

Dept. of Philosophy
University of Notre Dame
Notre Dame, IN 46556
U.S.A.